A Simple(ish) Derivation of the Heisenberg Uncertainty Principle

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The Heisenberg Uncertainty Principle relates the uncertainty in the position of a particle (Δx) and the uncertainty in its momentum (Δp):

$$\Delta p \Delta x \geq \frac{1}{2}\hbar$$

This states that the product of these two uncertainties (what you get when you multiply them together), must always be greater than, or at least equal to, some small numerical value ($\frac{\hbar}{2}$). The most important consequence of this is that neither of these uncertainties can ever be 0, in other words it is impossible to measure either the position or the momentum of a particle with perfect precision. What this also means is that, if we want to know the position or momentum to a high degree of precision we can, but this comes at the cost of greater uncertainty in the other value.

1 Derivation

Because of the probabilistic nature of quantum mechanics, the values of properties of the system that you can determine through a measurement or an experiment (these are called observables, and include things like the position, the energy or the momentum) can often take a range of values, and so it is convenient (and indeed fundamental to QM) that we work with average values for these. The average value for the property Ω (which could be anything you can measure by experiment) is represented in QM by the notation $\langle \Omega \rangle$. This is the same as the concept of average that you are familiar with from statistics.

Because QM is fundamentally probabilistic, there is a certain uncertainty associated with the average value of the observable. This is denotated $\Delta\Omega$ and is defined as follows

$$\Delta\Omega = \sqrt{\langle \Omega^2 \rangle - \langle \Omega \rangle^2}$$

This is the same as the standard deviation commonly used in statistics.

The final concept that we need to introduce is commutativity. Commutativity is an important property in mathematics, which basically defines whether changing the order of things matters. For example, in multiplication the order does not matter, because 3×4 is the same as 4×3 so it is said that multiplication commutes. Subtraction does not commute, because $3 - 4 \neq 4 - 3$. In QM, a commutator is defined as

$$[A, B] = AB - BA$$

which just means do B first and then A, then do A first then B, and subtract them. If A and B commute (like multiplication) then the answer will be 0.

We are now ready to see how the Heisenberg Uncertainty Principle is arrived at. This 'proof' cuts a lot of corners (mostly the difficult algebra), and relies on you accepting a few key results as being true, but hopefully can serve as a very gentle guide for the reader to see how the relation might appear.

A crucial result, that I am going to ask you to accept as being true, is the Robertson uncertainty relation, which relates the variances (simply the uncertainty squared, $\Delta\Omega^2$) of two observables (here I will call them Ω and K):

$$\Delta\Omega^2 \Delta K^2 \geq (\frac{1}{2i} \langle[\Omega, K]\rangle)^2$$

Lets unpack this. This is saying that the square of the uncertainties of two different observables multiplied together is always larger than, or equal to $\frac{1}{2i}$ (which is just a number, in the same way that 1 or 7 is) multiplied by the average value of the commutator of the two observables (the average value of $\Omega K - K\Omega$), all squared. This looks very threatening, but lets treat it as just a mathematical expression that we are going to use, like the equation for the energy levels in a particle in a box.

The other thing I need you to accept as true is a very famous result in QM, namely the commutator of position and momentum $[x, p] = i\hbar$. The average value of $i\hbar$, which is just a number, is $i\hbar$

So now, let's put the position (x) and the momentum (p) into the Robertson uncertainty relation:

$$\Delta x^2 \Delta p^2 \geq (\frac{1}{2i} \langle[x, p]\rangle)^2$$

putting the commutator result above into this gives

$$\Delta x^2 \Delta p^2 \geq (\frac{1}{2i} \times i \times \hbar)^2$$

which simplifies to

$$\Delta x^2 \Delta p^2 \geq (\frac{1}{2} \times \hbar)^2$$

Finally, taking the square root of both sides gives

$$\Delta x \Delta p \geq \frac{1}{2}\hbar$$

Look familiar!?